DESIGN OF ELECTRICAL MODELS FOR SOLVING THE HYPERBOLIC HEAT EQUATION

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The hyperbolic equation of heat conduction is solved on electrical models with lumped-parameter elements. Fundamental relations are established for the design of such models and the design procedure is described.

In the case of transient thermal processes occurring at high rates, the heat transmission is described more accurately by the hyperbolic than by the parabolic heat equation. The reason is that heat travels not infinitely fast, rather at some very high but finite velocity. Taking into account a finite velocity of heat propagation leads to the hyperbolic heat equation and it is absolutely necessary in an analysis of transient processes occurring at a high rate [1].

A solution of the nonlinear hyperbolic equation of heat transmission presents certain difficulties not easily overcome, especially in the case of complex and variable constraints. The use of electrical models with lumped-parameter elements may be helpful for the solution of this hyperbolic heat equation.

Most attractive are electrical models designed with resistances, capacitances, and inductances, since they simulate the process continuously in time and, therefore, are high-speed devices [2].

We will consider the asymmetric heating (cooling) of an anisotropic solid which interacts with the ambient medium under boundary conditions of the third kind. The transient heat transmission can be described mathematically as follows:

$$c\gamma\frac{\partial T}{\partial\tau}+c\gamma\tau_r\frac{\partial^2 T}{\partial\tau^2}=\operatorname{div}(\lambda_i\operatorname{grad}T),$$

$$T=T_N, \tag{1}$$

$$\frac{\partial T}{\partial i}+\frac{\alpha_n}{\lambda_i}(T_n-T)=0,$$

where

$$\operatorname{div}(\lambda_i\operatorname{grad}T)=\frac{\partial}{\partial x}\left(\lambda_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda_z\frac{\partial T}{\partial z}\right),$$

$$i=x,\ y,\ z;\quad n=\text{г},\ \text{в},\ \text{с},\ \text{вт},\ \text{рт},\ \text{zт}.$$

Following the theory of generalized variables, we transform the variables here (for the given process):

$$T=T_1\theta;\quad x=x_1X;\quad y=y_1Y;\quad z=z_1Z;\quad \tau=\tau_1t;$$

$$c=c_1C;\quad \gamma=\gamma_1\Gamma;\quad \lambda_x=\lambda_{1x}\Lambda_x;\quad \lambda_y=\lambda_{1y}\Lambda_y;\quad \lambda_z=\lambda_{1z}\Lambda_z,$$

where T_1, x_1, y_1, z_1, τ_1, c_1, γ_1, λ_{1x}, λ_{1y}, λ_{1z} are respectively the reference values of the temperature, the space coordinates, time, the specific heat, the density, and the thermal conductivities along the coordinate axes, while θ, X, Y, Z, t, C, Γ, Λ_x, Λ_y, Λ_z are the relative temperature, coordinates, time, specific heat, density, and thermal conductivities along the coordinate axes.

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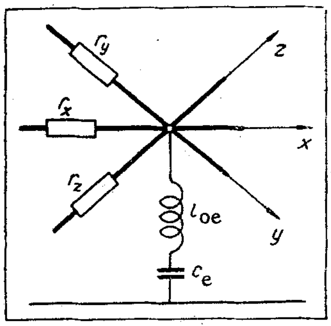

Fig. 1. Diagram of cell in electric space model.

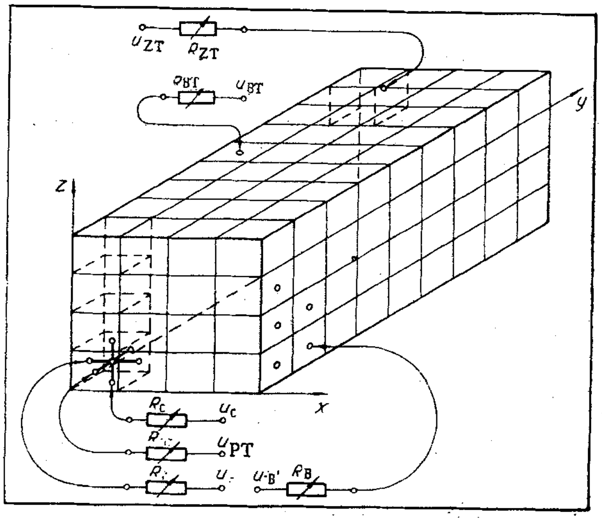

Fig. 2. Schematic diagram of an electrical model.

As a result, we obtain a mathematical model of the thermal process*

$$C\Gamma\left(\frac{\partial\theta}{\partial t}+A_1\frac{\partial^2\theta}{\partial t^2}\right)=A_2\frac{\partial}{\partial X}\left(\Lambda_x\frac{\partial\theta}{\partial X}\right)+A_3\frac{\partial}{\partial Y}\left(\Lambda_y\frac{\partial\theta}{\partial Y}\right)+A_4\frac{\partial}{\partial Z}\left(\Lambda_z\frac{\partial\theta}{\partial Z}\right),$$

$$\theta=A_5, \tag{2}$$

$$\frac{\partial\theta}{\partial I}+\frac{A_m}{\Lambda_i}(\theta_n-\theta)=0,$$

where m = 6, 7, 8, 9, 10, 11, and I = X, Y, Z.

Here coefficients A_1–A_{11} play the role of generalized parameters and they are defined as follows:

$$A_1=\frac{\tau_r}{\tau_1};\quad A_2=\frac{\lambda_{1x}\tau_1}{c_1\gamma_1x_1^2};\quad A_3=\frac{\lambda_{1y}\tau_1}{c_1\gamma_1y_1^2};\quad A_4=\frac{\lambda_{1z}\tau_1}{c_1\gamma_1z_1^2};$$

$$A_5=\frac{T_N}{T_1};\quad A_6=\frac{\alpha_\text{г}}{\lambda_{1x}}x_1;\quad A_7=\frac{\alpha_\text{в}}{\lambda_{1x}}x_1;\quad A_8=\frac{\alpha_c}{\lambda_{1z}}z_1; \tag{3}$$

$$A_9=\frac{\alpha_{ZT}}{\lambda_{1z}}z_1;\quad A_{10}=\frac{\alpha_{PT}}{\lambda_{1y}}y_1;\quad A_{11}=\frac{\alpha_{ZT}}{\lambda_{1y}}y_1.$$

For an isotropic medium and constant thermophysical properties (linear formulation of the problem), the mathematical model of the thermal process is

$$\frac{\partial\theta}{\partial t}+A_1\frac{\partial^2\theta}{\partial t^2}=A_2'\frac{\partial^2\theta}{\partial X^2}+A_3'\frac{\partial^2\theta}{\partial Y^2}+A_4'\frac{\partial^2\theta}{\partial Z},$$

$$\theta=A_5, \tag{4}$$

$$\frac{\partial\theta}{\partial I}+A_m'(\theta_n-\theta)=0,$$

where

$$A_2'=\frac{a\tau_1}{x_1^2};\quad A_3'=\frac{a\tau_1}{y_1^2};$$

$$A_4'=\frac{a\tau_1}{z_1^2};\quad A_6'=\frac{\alpha_\text{г}}{\lambda}x_1;\quad A_7'=\frac{\alpha_\text{в}}{\lambda}x_1;\quad A_8'=\frac{\alpha_c}{\lambda}z_1; \tag{5}$$

$$A_9'=\frac{\alpha_\text{вт}}{\lambda}z_1;\quad A_{10}'=\frac{\alpha_{PT}}{\lambda}y_1;\quad A_{11}'=\frac{\alpha_{ZT}}{\lambda}y_1.$$

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* By a mathematical model is meant a complete mathematical description of the process (including also the uniqueness conditions) in generalized variables.

The electrical transient process in a three-dimensional electrical model made up to resistances r, capacitances c_e, and inductances l_{oe} (Fig. 1) connected in series along the coordinate axes is mathematically described by the following system of equations*

$$
\begin{gathered}
c_e\left(\frac{\partial u}{\partial \tau_e}+\frac{l_{oe}}{r_e}\,\frac{\partial^2 u}{\partial \tau_e^2}\right)=\operatorname{div}\left(\frac{1}{r_i}\operatorname{grad} u\right),\\
u=u_N,\\
\frac{\partial u}{\partial i_e}+\frac{r_i}{R_n}(u_n-u)=0.
\end{gathered}
\tag{6}
$$

Following the theory of generalized variables and the analogy with the thermal process, through linear transformations we replace the process variables in system (6) by their generalized values.

As a result, we have a mathematical model of the electrical process

$$
\begin{gathered}
C_e\left(\frac{\partial U}{\partial t_e}+B_1\frac{L_{oe}}{R_e}\,\frac{\partial^2 U}{\partial t_e^2}\right)=B_2\frac{\partial}{\partial X_e}\left(\frac{1}{R_x}\cdot\frac{\partial U}{\partial X_e}\right)+B_3\frac{\partial}{\partial Y_e}\left(\frac{1}{R_y}\cdot\frac{\partial U}{\partial Y_e}\right)+B_4\frac{\partial}{\partial Z_e}\left(\frac{1}{R_z}\cdot\frac{\partial U}{\partial Z_e}\right),\\
U=B_5,\\
\frac{\partial U}{\partial I_e}+B_m R_i(U_n-U)=0,
\end{gathered}
\tag{7}
$$

where $I_e = X_e,\ Y_e,\ Z_e$.

The generalized parameters B_1–B_{11} of the electrical process are defined as follows:

$$
\begin{gathered}
B_1=\frac{l_{oe1}}{r_{1e}\tau_{1e}};\quad B_2=\frac{\tau_{1e}}{r_{1x}c_{1e}x_{1e}^2};\\
B_3=\frac{\tau_{1e}}{r_{1y}c_{1e}y_{1e}^2};\quad B_4=\frac{\tau_{1e}}{r_{1z}c_{1z}z_{1e}^2};\\
B_5=\frac{u_N}{u_1};\quad B_6=\frac{r_{1x}}{R_{\text{г}}}x_{1e};\\
B_7=\frac{r_{1x}}{R_{\text{в}}}x_{1e};\quad B_8=\frac{r_{1z}}{R_c}z_{1e};\\
B_9=\frac{r_{1z}}{R_{\text{вт}}}z_{1e};\quad B_{10}=\frac{r_{1y}}{R_{PT}}y_{1e};\quad B_{11}=\frac{r_{1y}}{R_{ZT}}y_{1e}.
\end{gathered}
\tag{8}
$$

The reference value r_{1e} can be determined from the relation

$$
\frac{1}{r_{1e}}\approx\frac{1}{3}\left(\frac{1}{r_{1x}}+\frac{1}{r_{1y}}+\frac{1}{r_{1z}}\right).
\tag{9}
$$

For an electrical model with constant parameters (resistances, capacitances, inductances), the mathematical model of the electrical process is

$$
\begin{gathered}
C_eR\left(\frac{\partial U}{\partial t_e}+B_1'\frac{L_{oe}}{R}\,\frac{\partial^2 U}{\partial t_e^2}\right)=B_2'\frac{\partial^2 U}{\partial X_e^2}+B_3'\frac{\partial^2 U}{\partial Y_e^2}+B_4'\frac{\partial^2 U}{\partial Z_e^2},\\
U=B_5,\\
\frac{\partial U}{\partial I_e}+B_m' R_i(U_n-U)=0,
\end{gathered}
\tag{10}
$$

where

$$
\begin{gathered}
B_1'=\frac{l_{oe1}}{r_1\tau_{1e}};\quad B_2'=\frac{\tau_{1e}}{r_1c_{1e}x_{1e}^2};\\
B_3'=\frac{\tau_{1e}}{r_1c_{1e}y_{1e}^2};\quad B_4'=\frac{\tau_{1e}}{r_1c_{1e}z_{1e}^2};\\
B_6'=\frac{r_1}{R_{\text{г}}}x_{1e};\quad B_7'=\frac{r_1}{R_{\text{в}}}x_{1e};
\end{gathered}
$$

* Additional resistances R_n are hooked on at the model boundaries.

$$B'_8 = \frac{r_1}{R_c} z_{1e}; \quad B'_9 = \frac{r_1}{R_{BT}} z_{1e};$$

$$B'_{10} = \frac{r_1}{R_{PT}} y_{1e}; \quad B'_{11} = \frac{r_1}{R_{ZT}} y_{1e}. \tag{11}$$

Since the mathematical models of the transient thermal process (Eqs. (2), (4)) and of the transient electrical process (Eqs. (7), (10)) both have the same structure, hence it is feasible to simulate on the basis of direct analogy.

Mathematical simulation requires also a sufficient degree of congruence between the mathematical model of the original and of the model. Inasmuch as the structures of both mathematical models are the same, it is necessary for simulation that the generalized parameters be respectively identical, i.e.,

$$A_1 = B_1; \quad A_2 = B_2; \quad A_3 = B_3; \; \ldots; \; A_{11} = B_{11}. \tag{12}$$

Equalities (12) are essential for designing electrical models on the basis of direct analogy, because they establish the quantitative relations between the electrical and the thermal process parameters.

Inserting relations (3), (5) and (8), (11) into equalities (12), we obtain the fundamental equations for the design of electrical models. In the nonlinear problem for the case of an anisotropic medium these equations become:

$$r_{Nx} = \frac{k_c \delta_x^2}{\lambda_{Nx} k_\tau n_x^2}; \quad R_c = \frac{\lambda_{Nz} r_{Nz}}{\alpha_c k_{lz}}; \quad r_{iy} = \frac{k_c \delta_y^2}{\lambda_{Ny} k_\tau n_y^2}; \quad R_{BT} = \frac{\lambda_{Nz} r_{Nz}}{\alpha_{BT} k_{lz}},$$

$$r_{PZ} = \frac{k_c \delta_z^2}{\lambda_{Nz} k_\tau n_z^2}; \quad R_{PT} = \frac{\lambda_{Ny} r_{Ny}}{\alpha_{PT} k_{ly}}; \quad R_\Gamma = \frac{\lambda_{Nx} r_{Nx}}{\alpha_\Gamma k_{lx}}; \quad R_{ZT} = \frac{\lambda_{Ny} r_{Ny}}{\alpha_{ZT} k_{ly}}; \tag{13}$$

$$R_{Nx} = \frac{\lambda_{Nx} r_{Nx}}{\alpha_B k_{lx}}; \quad l_{oe} = \frac{r_{1e} \tau_r}{k_\tau}.$$

A change to limiting values yields scale factors for the temperature (k_T), the coordinates (k_l), time (k_τ), the thermal conductivity (k_λ), and the capacitance (k_C):

$$k_T = \frac{T_{max}}{u_{max}}; \quad k_{lx} = \frac{\delta_x}{n_x}; \quad k_{ly} = \frac{\delta_y}{n_y}; \quad k_{lz} = \frac{\delta_z}{n_z}; \quad k_\tau = \frac{\tau_T}{\tau_e},$$

$$k_{\lambda x} = \lambda_{Nx} r_{Nx}; \quad k_{\lambda y} = \lambda_{Ny} r_{Ny}; \quad k_{\lambda z} = \lambda_{Nz} r_{Nz}; \quad k_c = \frac{c_N \gamma_N}{c_e}. \tag{14}$$

From the system of design equations one can obtain criterial numbers for the heat transmission asymmetry [3] along the coordinate axes:

$$\sigma_x = \frac{\alpha_\Gamma}{\alpha_B} = \frac{R_B}{R_\Gamma}; \quad \sigma_y = \frac{\alpha_{PT}}{\alpha_{ZT}} = \frac{R_{ZT}}{R_{PT}}; \quad \sigma_z = \frac{\alpha_c}{\alpha_{BT}} = \frac{R_{BT}}{R_c}. \tag{15}$$

In the design of electrical models all thermal and structural parameters of the solid body must be already known. Only the parameters of the electrical model are still to be determined. The ten equations (13) for the design of electrical models contain 15 unknowns: r_{Nx}, r_{Ny}, r_{Nz}, c_e, n_x, n_y, n_z, R_Γ, R_B, R_C, R_{BT}, R_{PT}, R_{ZT}, l_{oe}, k_T. In order to determine them, one must fix five parameters. One must also bear in mind then that the scale factors for the temperature, the coordinates, the conductivity, and the capacity are defined by relations (14). If n_x, n_y, n_z, c_e, and k_T are given, for example, then first the resistances of the model elements r_{Nx}, r_{Ny}, r_{Nz} are found from relations (13) followed by the boundary-value resistances R_Γ, R_B, R_C, R_{BT}, R_{PT}, R_{ZT}, and inductance l_{oe}. A model was constructed with the parameters thus found. In Fig. 1 is shown one cell of a three-dimensional electrical model. The circuit of the electrical model is shown in Fig. 2. On the same diagram is also shown the electrical circuit of the boundary-value resistances.

If the material is isotropic and its thermophysical properties can be assumed constant and equal to their mean values over the operating temperature range, then the system of design equations becomes

$$c_e = \frac{\delta_x^2}{a r k_\tau n_x^2}; \quad R_c = \frac{\lambda r n_z}{\alpha_c \delta_z}; \quad r = \frac{\delta_y^2}{a c_e k_\tau n_y^2}; \quad R_{BT} = \frac{\lambda r n_z}{\alpha_{BT} \delta_z};$$

$$n_z^2 = \frac{\delta_z^2}{arc_e k_\tau}; \quad R_{PT} = \frac{\lambda r n_y}{\alpha_{PT}\delta_y}; \quad R_\Gamma = \frac{\lambda r n_x}{\alpha_\Gamma \delta_x}; \quad R_{ZT} = \frac{\lambda r n_y}{\alpha_{ZT}\delta_y};$$

$$R_B = \frac{\lambda r n_x}{\alpha_B \delta_x}; \quad l_{oe} = \frac{r\tau_r}{k_\tau}. \tag{16}$$

Equations (16) contain 13 unknown parameters of the electrical model: r, ce, n_x, n_y, n_z, R_Γ, R_B, R_C, R_{BT}, R_{PT}, R_{ZT}, l_{oe}, k_τ. In order to determine them, one must fix three quantities. One must also bear in mind that the fixed quantities should simultaneously include three which are rigidly coupled by one of the equations in system (16). The appropriate design variant is selected on the basis of its assembly feasibility.

The system of Eqs. (13) or (16) is used for designing models as well as for calculating the steady-state parameters during simulation. Operating experience with electrical models made up of resistances, capacitances, and inductances has shown that they can be successfully used for the solution of practical heat engineering problems.

NOTATION

T	is the temperature;
c	is the specific heat;
γ	is the density;
λ	is the thermal conductivity;
a	is the thermal diffusivity;
α	is the heat transfer coefficient;
x, y, z	are the space coordinates in the medium;
A, B, A', B'	are the coefficients playing the role of generalized parameters;
τ	is the time;
τ_r	is the relaxation time;
θ	is the relative temperature;
X, Y, Z	are the relative coordinates;
t	is the relative time;
C	is the relative heat capacity;
Γ	is the relative density;
Λ	is the relative thermal conductivity;
c_e	is the cell capacitance of the electrical model;
r_e	is the cell resistance of the electrical model;
R	is the boundary resistance of the electrical model;
l_{oe}	is the inductance of the electrical model;
U	is the relative voltage;
X_e, Y_e, Z_e	are the relative coordinates of the electrical model;
L_{Oe}	is the relative cell inductance of the electrical model;
R_e	is the relative cell resistance of the electrical model;
R_x, R_y, R_z	are the relative cell resistances along the coordinate axes;
C_e	is the relative cell capacitance of the electrical model;
k_T	is the temperature scale;
k_τ	is the time scale;
k_l	is the coordinate scale;
k_λ	is the scale of thermal conductivity;
k_c	is the scale of specific heat;
n	is the number of cells in the electrical model;
δ	is the linear dimension of the solid body;
σ	is the criterion of heat transfer asymmetry.

Subscripts

N	denotes the initial-state parameter value;
i	denotes the coordinates (x, y, z);
n	denotes the media interacting with the solid body;
e	denotes the electrical process or the parameters of the electrical model;
1	denotes the reference value.

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